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# The structure analysis of fuzzy sets

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**Abstract**

Since the naissance of the fuzzy set theory in 1965, it has been applied to many areas extensively. In the applications, people are confronted with the interpretation problem of membership functions frequently. Since each person has his/her own opinion about the meaning of a subjective concept, he/she always has his/her own membership function for the same concept. The applications of the theory, for example fuzzy control and fuzzy reasoning, show the robustness of the (more or less) optionally chosen membership functions. This phenomenon probably reflects the inherent characteristics of fuzzy sets. In order to uncover the reason, many researchers have pay attention to the interpretation of membership functions (fuzzy sets). In this paper, from the quotient space theory and fuzzy equivalence relation, a new structural definition of fuzzy set is given. The "isomorphic principle" and "similarity principle" of fuzzy sets, the necessary and sufficient conditions of the "isomorphism" and "$\varepsilon$-similarity" of two fuzzy equivalence relations, and some properties of the new structural definition are discussed. These results may open up some inherent properties of fuzzy sets and provide a new interpretation of the membership functions.


*Keywords:* Fuzzy set; Membership function; Structural definition of fuzzy sets; Quotient space theory; Isomorphic principle; Similarity principle

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## 1. Introduction

Since the naissance of the fuzzy set theory in 1965 [1], it has been applied to many areas extensively. In the applications, people are confronted with the interpretation problem of the membership functions frequently. Since each person has his/her own opinion about the meaning of the same subjective concept such as "small", "large", etc., he/she always has his/her own membership function for the same concept such as triangular, trapezoidal, or gaussian, to name a few. The applications of the theory, for example fuzzy control and fuzzy reasoning, show the robustness of the (more or less) optionally chosen membership functions. This phenomenon probably reflects the inherent characteristics of fuzzy sets. In order to uncover the reason, we will give a new interpretation to the membership functions.

The definition of the membership function as follows

**Definition 1.1.** Given a domain $X$. If $\underline{A}$ is a fuzzy subset of $X$, for any $x \in X$ assigning a number $\mu_A(x)$: $X \to [0,1]$ to $x$, $\mu_A(x)$ is called a membership of $x$ with respect to $\underline{A}$.

Mapping $\mu_A(x){:}X \to [0,1], x \to \mu_A(x)$ is called a membership function of $\underline{A}$.

It is noted that in the following discussion, domain $X$ is assumed to be infinite (not limit to finite). For simplicity, the membership function is denoted by $A(x)$ rather than $\mu_A(x)$.

The main operations of fuzzy sets: given two fuzzy sets $A(x)$, $B(x)$, the union, intersection and complement of $A(x)$ and $B(x)$ are

$$(A \cup B)(x) = \max[A(x), B(x)], \quad (A \cap B)(x) = \min[A(x), B(x)], \quad \overline{A} = [1 - A(x)]$$

In practice such as fuzzy reasoning, designers may choose the membership functions optionally in some degree, i.e., these membership functions may (slightly) be different for the same concept, but they can generally get the same (or approximate) result. The robustness of the (more or less) optionally chosen membership functions has brought many people's attention. Some researchers [2–5] presented the probabilistic interpretation of membership functions. For example, Lin [2,3] interpreted memberships as probabilities. Each sample space has a probability and each point is associated with one sample space. So the total space is like a fibre space. Each fibre space is a probability space. Liang [4] regarded the membership function value as an independent and identically distributed random variable and proved that the mean of the membership functions exists for all the elements of the universe of discourse. He interpreted the meaning of a subjective concept of a group of people as the mean of the membership function for all people within the group. Takashi Mitsuishi [5] introduced a new concept of empty fuzzy set in order to define the membership function of fuzzy sets. These results may interpret the robustness of membership functions in the probabilistic sense. That is, although different persons may assign (slightly) different membership functions to a concept when solving a real problem (fuzzy control or fuzzy reasoning), in average they can get an approximate result. Unfortunately, these results were obtained based on a strong assumption, i.e., the membership function value is assumed to be an independent and identically distributed random variable. Lin [6–8] presented a topological definition, topological rough set, of fuzzy sets by using neighborhood systems, discussed the properties of fuzzy sets from their structure, and then presented a definition of the equivalence between two fuzzy membership functions and the necessary and sufficient conditions of the equivalence between two membership functions. He also discussed the concept of granular fuzzy sets in [2,9] and "elastic" membership functions in [10–12]. Lin's works provide a structural interpretation of membership functions (fuzzy sets).

It can be seen that a membership function of fuzzy sets can be interpreted in two ways: one probabilistic, the other structural. We will show below that for a fuzzy set (concept), it may probably be described by different types of membership functions, as long as their structures (see the structural definition of fuzzy sets below) are the same, they will represent the fuzzy sets with the same property. That is, although these membership functions are different in appearance, they are the same in essence. Therefore, the structural interpretation of fuzzy sets would be better than the probabilistic one. And it seems that in a given environment most people would have a similar structural interpretation for a concept. We will introduce a structural definition of fuzzy sets and discuss its properties below since the structural description is more essential to a fuzzy set.

The paper is organized as follows. In Section 2, the structural definition of membership functions and the isomorphism principle are discussed. In Section 3, the equivalence discrimination of membership functions is presented. In Section 4, the definition of ε-similarity and the similarity principle are introduced. Section 5, some main conclusions are made. First, we introduce some basic results of quotient space theory [13–15].

**Definition 1.2.** Assume $R \in T(X \times X)$, where $T(X \times X)$ stands for all fuzzy sets on $X \times X$. If it satisfies
(1) $\forall x,\ R(x,x) = 1$,
(2) $\forall x, y \in X,\ R(x,y) = R(y,x)$,
(3) $\forall x, y, z \in X,\ R(x,z) \geqslant \sup_y(\min(R(x,y), R(y,z)))$,

$R$ is called a fuzzy equivalence relation on $X$.

**Proposition 1.1.** *Assume that $R$ is a fuzzy equivalence relation on $X$. If we define $\forall x, y \in X,\ x \sim y \Longleftrightarrow R(x,y) = 1$, then "$\sim$" is a crisp equivalence relation on $X$. Let $[X]$ be a quotient space corresponding to relation "$\sim$", i.e.,*

$$[X] = \{[x] | x \in X\}, [x] = \{y | y \sim x, y \in X\}.$$

**Proof.** Self-reflectance and symmetry properties are obvious. Now we prove the transitivity. From $x \sim y$, $y \sim z \Rightarrow R(x,y) = 1, R(y,z) = 1$, have $R(x,z) \geqslant \min(R(x,y), R(y,z)) = 1$ , i.e., $x \sim z$. □

**Theorem 1.1.** *Assume that* $R$ *is a fuzzy equivalence relation on* $X$, *and* $[X]$ *is a quotient space as defined in Proposition* 1.1. *Define*

$$\forall a, b \in [X], d(a,b) = 1 - R(x,y) \quad \forall x \in a, y \in b, \tag{1.1}$$

$d(\cdot,\cdot)$ *is a distance function on* $[X]$.

**Proposition 1.2.** *Assume that* $R$ *is a fuzzy equivalence relation on* $X$. *Let* $R_\lambda = \{(x,y)|R(x,y) \geqslant \lambda\}$, $0 \leqslant \lambda \leqslant 1$, $R_\lambda$ *is a crisp equivalence relation on* $X$ *and is called a cut relation of* $R$.

**Proof.** Obviously, relation $R_\lambda$ has self-reflectance and symmetry properties. Assume that $x$ and $y$, and $y$ and $z$ both are $R_\lambda$ equivalence. From its definition, we have $R(x,y) \geqslant \lambda$ and $R(z,y) \geqslant \lambda$. From (3) in Definition 1.2, have $R(x,z) \geqslant \min\{R(x,y), R(y,z)\} \geqslant \lambda$, i.e., $R(x,z) \geqslant \lambda$. Then $x$ and $z$ are $R_\lambda$ equivalence. Relation $R_\lambda$ has transitivity property. Finally, $R_\lambda$ is a crisp equivalence relation. □

From the definition of $R_\lambda$, it is noted that $0 \leqslant \lambda_2 \leqslant \lambda_1 \leqslant 1 \Longleftrightarrow R_{\lambda_1} > R_{\lambda_2} \Longleftrightarrow X(\lambda_2)$ is a quotient space of $X(\lambda_1)$, where $X(\lambda)$ is a quotient space corresponding to equivalence relation $R_\lambda$. A family $\{X(\lambda)|R(x,y) \geqslant \lambda\}$, $0 \leqslant \lambda \leqslant 1$, of quotient spaces forms an order-chain based on the inclusion relation of quotient sets. A set $\{X(\lambda)|0 \leqslant \lambda \leqslant 1\}$ of spaces is called a hierarchical structure on $X$. Therefore, given a fuzzy equivalence relation $R$ on $X$, we have a corresponding hierarchical structure on $X$.

Below we will show in what condition a corresponding fuzzy equivalence relation can be obtained from a normalized metric.

**Definition 1.3.** Assume that $(X,d), d(x,y) \leqslant 1$, is a metric space. For $\forall x,y,z \in (X,d)$, if there does not exist any number within the array $\{d(x,y), d(y,z), d(x,z)\}$ such that it is greater than other two numbers, $d$ is called a normalized equicrural distance.

From the definition, it is known that the necessary and sufficient condition that $d$ is a normalized equicrural distance is that any triangle formed by connecting any three points on $X$ is an equicrural triangle and its crus is the longest side of the triangle. Since a triangle formed by $\{d(x,y), d(y,z), d(x,z)\}$ is equicrural and its crus is the longest side of the triangle, there must exist two equal numbers within $\{d(x,y), d(y,z), d(x,z)\}$ such that the two numbers do not less than the third one. That is, there does not exist any number within $\{d(x,y), d(y,z), d(x,z)\}$ such that it is greater than other two numbers. The name "equicrural distance" is used for indicating its geometric meaning.

**Proposition 1.3.** *Assume that* $\{X(\lambda)|0 \leqslant \lambda \leqslant 1\}$ *is a hierarchical structure on* $X$. *There exists a fuzzy equivalence relation* $R$ *on* $X$ *such that* $X(\lambda)$, $\lambda \in [0,1]$, *is a quotient space corresponding to* $R_\lambda$, *and* $R_\lambda$ *is a cut relation of* $R$.

**Proof.** Since $\{X(\lambda)\}$ is a hierarchical structure on $X$, each $X(\lambda)$ is a quotient space of $X$. Let $R_\lambda$, $0 \leqslant \lambda \leqslant 1$, be an equivalence relation corresponding to $X(\lambda)$.

$\forall x, y \in X$, define

$$R(x,y) = \begin{cases} \inf\{\lambda|(x,y) \notin R_\lambda\} \\ 1, \quad \forall\lambda, (x,y) \in R_\lambda \end{cases}$$

$\forall x,y,z \in X$, Let $R(x,y) = a_1$, $R(x,z) = a_2$, $R(y,z) = a_3$.

For $\forall\varepsilon > 0$, we have $a_i - \varepsilon < d_1 \leqslant a_1$, $a_2 - \varepsilon < d_2 \leqslant a_2$, $d_3 < a_3 - \varepsilon < d_3 \leqslant a_3$.

$(x,y) \in R_{d1}$, $(x,z) \in R_{d2}$, $(y,z) \in R_{d3}$.

If $d_2 \geqslant \min(d_1,d_3)$, then $R(x,z) \geqslant d_2 \geqslant \min(R(x,y) - \varepsilon, R(y,z) - \varepsilon) \geqslant \min(R(x,y), R(y,z)) - \varepsilon$.

If $d_2 < \min(d_1,d_3)$, assuming $d_3 \leqslant d_1$, from $(x,y) \in R_{d1}$, $(y,z) \in R_{d3}$, we have $(x,y) \in R_{d3}$, i.e., $x \sim y$, $y \sim z(R_{d3})$, then $x \sim z(R_{d3})$, i.e., $R(x,z) \geqslant d_3 = \min(d_1,d_3) \geqslant \min(R(x,y), R(y,z)) - \varepsilon$.

Let $\varepsilon \to 0$, we have

$R(x,z) \geqslant \sup_y(\min(R(x,y), R(y,z)))$.

$R(x,y)$ is a fuzzy equivalence relation on $X$ and $R_\lambda$ is its cut relation. □

**Theorem 1.2.** *The following three statements are equivalent*, *i.e.*,

(1) *Given a fuzzy equivalence relation on X.*
(2) *Given a normalized equicrural distance on some quotient space of X.*
(3) *Given a hierarchical structure on X.*

**Proof.** (1) → (2), it is needed to show that the distance defined on $[X]$ based on Theorem 1.1 is a normalized equicrural distance. By reduction to absurdity, assume that there exist $a,b,c \in ([X],d)$ such that within $\{d(a,b) = d_1,\ d(b,c) = d_2,\ d(a,c) = d_3\}$ there is a number greater than other two numbers, e.g., $d_3 > d_2$, $d_3 > d_1$, $d_2 \geqslant d_1$. From the condition (3) of Definition 1.2, we have $R(a,c) \geqslant \min\{R(a,b), R(b,c)\}$, i.e., $1 - d_3 \geqslant \min\{1 - d_1, 1 - d_2\} = 1 - d_2$. Then have $-d_3 \geqslant -d_2$, i.e., $d_3 \leqslant d_2$. This contradict to $d_3 > d_2$.

(2) → (1), assume that $d(a,b)$ is a normalized equicrural distance on $[X]$. When $a,b \in [X]$, let $R(a,b) = 1 - d(a,b)$. For $x,y \in X$, $x \in a$, $y \in b$, $a,b \in [X]$, define $R(x,y) = R(a,b)$. We will show that $R(x,y)$ is a fuzzy equivalence relation on $X$.

Obviously, $R(x,y)$ has self-reflectance and symmetry properties. Now we show its transitivity property. By reduction to absurdity, there exist $x,y,z \in X$, $x \in a$, $y \in b$, $z \in c$, $a,b,c \in [X]$ such that $R(x,z) < \min\{R(x,y), R(y,z)\}$. Assume that $d(a,b) \geqslant$

$d(b,c)$. Then $R(a,c) < \min\{R(a,b), R(b,c)\}$. We have $1 - d(a,c) < \min\{1 - d(a,b), 1 - d(b,c)\} = 1 - d(a,b)$, then $d(a,c) > d(a,b) \geqslant d(b,c)$. This is in contradiction to that $d$ is a normalized equicrural distance.

(1) → (3), it can be obtained from Proposition 1.2.
(3) → (1), it can be obtained from Proposition 1.3. □

## 2. A structural definition of membership functions

**Definition 2.1.** Given a fuzzy equivalence relation $R(x,y)$ on $X$. For a crisp set $A$ on $X$, we define a corresponding fuzzy set $\underline{A}$ such that its membership function is $A(x) = \sup\{R(x,y)|y \in A\}$. $A(x)$ is called a structural definition of membership functions.

Therefore, $\underline{A}$ is a fuzzy set extended from a crisp set $A$ by fuzzy equivalence relation $R$ and with $A$ as its core. The new definition is induced from a fuzzy equivalence relation and represents the relationship between a crisp set and its corresponding fuzzy set so that it redounds to the understanding of fuzzy sets. We will show below that the new definition is reasonable.

Lin [6,7] defined fuzzy sets by neighborhood systems and discussed the operations of the newly defined fuzzy sets. Yao [16] presented a qualitative definition of fuzzy sets. The definition we proposed is quite similar to their definitions. But we will mainly focus on the relation between the operations of the fuzzy sets commonly defined and that of the fuzzy sets we proposed in order to uncover the essence of membership functions. First, we show that two different fuzzy equivalence relations may correspond to the same hierarchical structure (see Example 1).

**Example 1.** Assume $X = \{1,2,3,4\}$. Two fuzzy equivalence relations $R_1, R_2$ are as follows. Fuzzy equivalence relation $R_1$: $R_1(1,1) = R_1(2,2) = R_1(3,3) = R_1(4,4) = 1$; $R_1(1,2) = 0.8$; $R_1(1,3) = R_1(1,4) = R_1(2,3) = R_1(2,4) = 0.5$; $R_1(3,4) = 0.6$. Its corresponding hierarchical structure: $X_1(1) = \{1,2,3,4\}$, $X_1(0.8) = \{(1,2),3,4\}$, $X_1(0.6) = \{(1,2),(3,4)\}$, $X_1(0.5) = \{(1,2,3,4)\}$.

Fuzzy equivalence relation $R_2$: $R_2(1,1) = R_2(2,2) = R_2(3,3) = R_2(4,4) = 1$; $R_2(1,2) = 0.9$; $R_2(1,3) = R_2(1,4) = R_2(2,3) = R_2(2,4) = 0.6$; $R_2(3,4) = 0.7$. Its corresponding hierarchical structure: $X_2(1) = \{1,2,3,4\}$, $X_2(0.9) = \{(1,2),3,4\}$, $X_2(0.7) = \{(1,2),(3,4)\}$, $X_2(0.6) = \{(1,2,3,4)\}$.

From the example, it can be seen that $R_1$ and $R_2$ are two different fuzzy equivalence relations but they have the same hierarchical structure. In fact, from a hierarchical structure, a limitless number of fuzzy equivalence relations can be defined.

**Definition 2.2.** Fuzzy equivalence relations $R_1$ and $R_2$ having the same corresponding hierarchical structure are called isomorphic.

**Definition 2.3.** Assume that $R_1$ and $R_2$ are two fuzzy equivalence relations on $X$, and $\{X_1(\lambda)/0 \leqslant \lambda \leqslant 1\}$ and $\{X_2(\mu)/0 \leqslant \mu \leqslant 1\}$ are their corresponding hierarchical structures, respectively. If there exists an one-one corresponding and increasing function $f$ :$[0,1] \to [0,1]$ ($[0,1]$ onto $[0,1]$) such that $\forall \lambda$, $0 \leqslant \lambda \leqslant 1$ we have $X_1(\lambda) = X_2(f(\lambda))$, $R_1$ and $R_2$ are called as strongly isomorphic. Function $f$ is an isomorphic transform of $R_1$ and $R_2$.

Note that if two fuzzy equivalence relations are strongly isomorphic, they must be isomorphic. Contrarily, the relationship does not hold.

**Definition 2.4.** Given a fuzzy subset $\underline{A}$ and its membership function $\mu_A(x)$. Defining an equivalence relation $R : x \sim y \Longleftrightarrow \mu_A(x) = \mu_A(y)$ on $X$, we have a quotient space $[X]_A$ corresponding to $R$. Furthermore, defining an order "<" on $[X]_A$ such that $[x] < [y] \Longleftrightarrow \mu_A(x) \leqslant \mu_A(y)$, $x \in [x]$, $y \in [y]$, space $([X]_A, <)$ obtained is a totally ordered quotient space corresponding to fuzzy subset $\underline{A}$.

**Definition 2.5.** Fuzzy subsets $\underline{A}$ and $\underline{B}$ are isomorphic if they belong to a corresponding totally ordered quotient space.

**Proposition 2.1.** *Given two isomorphic fuzzy equivalence relations $R_1$, $R_2$ and a crisp set A. Fuzzy subsets $\underline{A}_1$ and $\underline{A}_2$ are isomorphic, where $\underline{A}_1$ and $\underline{A}_2$ are defined by $R_1$ and $R_2$ respectively according to Definition* 2.1 (*the structural definition*).

**Proposition 2.2.** *$R_1$ and $R_2$ are two strong isomorphic fuzzy equivalence relations on X. A and B are two crisp sets. Based on Definition* 2.1, *from $R_1$ and $R_2$ fuzzy subsets $A_1$, $\underline{A}_2$ and $\underline{B}_1$, $\underline{B}_2$ can be defined respectively. Then, fuzzy set $\underline{A}_1 \cup \underline{B}_1$ and fuzzy set $\underline{A}_2 \cup \underline{B}_2$ ($\underline{A}_1 \cap \underline{B}_1$ and $\underline{A}_2 \cap \underline{B}_2$) are isomorphic.*

**Proof.** $A_1(x)$, $A_2(x)$, $B_1(x)$, and $B_2(x)$ are membership functions correspond to $\underline{A}_1$, $\underline{A}_2$, $\underline{B}_1$, and $\underline{B}_2$, respectively. Assume that $C_1(x)$ and $C_2(x)$ are membership functions of $\underline{A}_1 \cup \underline{B}_1$ and $\underline{A}_2 \cup \underline{B}_2$ respectively. $C_1(x) = \max[A_1(x), B_1(x)]$, $C_2(x) = \max[A_2(x), B_2(x)]$.

Letting $I_{A1} = \{x|C_1(x) = A_1(x)\}$, $I_{A2}\{x|C_2(x) = A_2(x)\}$, we will show $I_{A1} = I_{A2}$ below.

If $x \in I_{A1}$, then $A_1(x) \geqslant B_1(x)$. From Proposition 2.1, $A_1(x)$ and $A_2(x)$, $B_1(x)$ and $B_2(x)$ are isomorphic. By assuming that $A_1(x) > B_1(x)$, from the definition of $A(x)$ there exists $y \in A$ such that $R_1(x,y) > A_1(x) - \varepsilon$. On the other hand, for $\forall z \in B$ have $R_1(x,z) \leqslant B_1(x)$. Letting $\varepsilon$ small enough such that $\forall z \in B$, $R_1(x,y) > R_1(x,z)$, since $R_1$ and $R_2$ are isomorphic, we have

$R_2(x,y) > R_2(x,z)$, $\forall z \in B$, (see the conclusion of Theorem 3.1).

By taking the upper bound of the right hand side of the above inequality,

$R_2(x,y) \geqslant \sup\{R_2(x,z), \forall z \in B\} = B_2(x)$. Then we have $A_2(x) \geqslant B_2(x)$, i.e., $x \in I_{A2}$.

Similarly, if $x \in I_{A2}$ then we have $x \in I_{A1}$. Finally, $I_{A1} = I_{A2}$.

Next, we show below that $C_1(x)$ and $C_2(x)$ are isomorphic.

Given $x$ and $y$, and $C_1(x) > C_1(y)$. If $x, y \in I_{A1}$, then $C_1(x) = A_1(x)$, $C_1(y) = A_1(y)$, and have $A_1(x) > A_1(y)$. From $I_{A1} = I_{A2}$, have $C_2(x) = A_2(x)$, $C_2\ (y) = A_2(y)$. Then, since $A_1$ and $A_2$ are isomorphic, $C_2\ (x) = A_2(x) > A_2(y) = C_2(y)$.

Similarly, we may have the same result when $x$ and $y$ both do not belong to $I_{A1}$.

Finally, we need to consider the instance: $x \in I_{A1}$ and $y \notin I_{A1}$.

Assume $x \in I_{A1}$, $y \notin I_{A1}$, and $C_1(x) > C_1(y)$, then $A_1(x) > B_1(y)$. Letting $\underline{A}_1(x) = \lambda_1$, $\underline{B}_1(y) = \lambda_2$, we have $\lambda_1 > \lambda_2$. Let $\lambda_1 > \lambda_3 > \lambda_4 > \lambda_2$, from $R_1$ and $R_2$ are strong isomorphic, i.e., $f$ is a isomorphic transform of $R_1$ and $R_2$, by letting $\mu_3 = f(\lambda_3)$ and $\mu_4 = f(\lambda_4)$, we have $\mu_4 < \mu_3$ and $X_1(\lambda_3) = X_2(\mu_3)$, $X_1(\lambda_4) = X_2(\mu_4)$.

From the definition of $A_1(x)$, we have $\exists z \in A$, $R_1(z,x) \geqslant \lambda_3$, i.e., $z$ and $x$ are equivalent on $X_1(\lambda_3)$. Thus, $x$ and $z$ are equivalent on $X_2(\mu_3)$ as well. We have $A_2(x) \geqslant R_2(z,x) \geqslant \mu_3$. Similarly, $y$ and any point in $B$ are not equivalent on $X_1(\lambda_4)$. The $y$ and any point in $B$ are not equivalent on $X_2(\mu_4)$, i.e., $B_2(y) \leqslant \mu_4 < \mu_3 \leqslant A_2(x)$. We have $C_2(y) = B_2(y) \leqslant \mu_4 < \mu_3 \leqslant A_2(x) = C_2(x)$, i.e., $C_2(y) < C_2(x)$.

Similarly, when $C_1(x) < C_1(y,) \Longleftrightarrow C_2(x) < C_2(y)$ and $C_1(x) = C_1(y,) \Longleftrightarrow C_2(x) = C_2(y)$, it can be proved that $\underline{A}_1 \cup \underline{B}_1$ and $\underline{A}_2 \cup \underline{B}_2$ are isomorphic.

Similarly, it can be proved that $\underline{A}_1 \cap \underline{B}_1$ and $\underline{A}_2 \cap \underline{B}_2$ are isomorphic. □

**Proposition 2.3.** *$R_1$ and $R_2$ are two isomorphic fuzzy equivalence relations on X. A is a crisp set on X. From $R_1$ and $R_2$, according to Definition* 2.1 *fuzzy subsets $\underline{A}_1$ and $\underline{A}_2$ can be defined. Then, fuzzy sets $\underline{A}_1^-$ and $\underline{A}_2^-$ are isomorphic, where $\underline{A}_1^-$ is the complement of $\underline{A}_1$ and its membership function is $A_1^-(x) = (1 - A_1(x))$.*

**Proof.** Since the corresponding totally ordered quotient spaces of fuzzy sets $\underline{A}_1^-$ and $\underline{A}$ are the same, (only the orders in the two spaces are reciprocal), $\underline{A}_1^-$ and $\underline{A}_2^-$ are isomorphic. □

**Theorem 2.1** (weak isomorphism principle). *$R_1$ and $R_2$ are two strongly isomorphic fuzzy equivalence relations on X. Given a family $\{A_1, A_2, A_n\}$ of crisp sets. From $R_1$ and $R_2$, the families $A = \{\underline{A}_1, \ldots, \underline{A}_n\}$ and $B = \{\underline{B}_1, \ldots, \underline{B}_n\}$ of fuzzy subsets can be defined. After performing a finite number of set operations* (*complement*, *intersection*, *union*, *etc.*) *over A and B*, *we have new families $C = \{\underline{C}_1, \ldots, \underline{C}_m\}$ and $D = \{\underline{D}_1, \ldots, \underline{D}_m\}$ of fuzzy subsets. C and D are isomorphic as well.*

**Proof.** The theorem can directly be obtained from Proposition 2.2 and 2.3. □

In Section 4, we will show that when the condition "strongly isomorphic" is replaced by "isomorphic", the conclusion of Theorem 2.1 still holds. Since the condition in Theorem 2.1 is strong, we call the theorem "weak".

The theorem shows that the totally ordered quotient space represents the inherent property of a fuzzy set. After performing a family of set operations over any two fuzzy sets of the same (isomorphic) structure, the newly obtained fuzzy sets still have the same structure whereas they may probably be described by different membership functions. The theorem would give a deeper understanding of the fuzzy set theory.

**Example 2.** Fuzzy equivalence relations $R_1$ and $R_2$ as defined in Example 1. Let $A = \{1,3\}$ and $B = \{1\}$. Define fuzzy sets $\underline{A}_1$, $\underline{B}_1$ and $\underline{A}_2$, $\underline{B}_2$ from $R_1$ and $R_2$, respectively. Their membership functions as follow:

$\underline{A}_1(x) = \{\underline{A}_1(1) = 1,\ \underline{A}_1(2) = 0.8,\ \underline{A}_1(3) = 1,\ \underline{A}_1(4) = 0.6\}$
$\underline{B}_1(x) = \{\underline{B}_1(1) = 1,\ \underline{B}_1(2) = 0.8,\ \underline{B}_1(3) = 0.5,\ \underline{B}_1(4) = 0.5\}$
$\underline{A}_2(x) = \{\underline{A}_2(1) = 1,\ \underline{A}_2(2) = 0.9,\ \underline{A}_2(3) = 1,\ \underline{A}_2(4) = 0.7\}$
$\underline{B}_2(x) = \{\underline{B}_2(1) = 1,\ \underline{B}_2(2) = 0.9,\ \underline{B}_2(3) = 0.6,\ \underline{B}_2(4) = 0.6\}$

Fuzzy sets $\underline{A}_1$ and $\underline{A}_2$ have different membership functions on $X$. But they are isomorphic since their corresponding totally ordered quotient spaces $(\{(1,3),(2),(4)\})$ are the same.

Similarly, fuzzy sets $(\underline{A}_1 \cap \underline{B}_1)$ and $(\underline{A}_2 \cap \underline{B}_2)$ have different membership functions, their corresponding totally ordered quotient spaces $(\{(1),(2),(3,4)\})$ are the same as well.

$$\begin{aligned}(\underline{A}_1 \cap \underline{B}_1)(x) = \{&(\underline{A}_1 \cap \underline{B}_1)(1) = 1, (\underline{A}_1 \cap \underline{B}_1)(2) = 0.8,\\ &(\underline{A}_1 \cap \underline{B}_1)(3) = 0.5, (\underline{A}_1 \cap \underline{B}_1)(4) = 0.5\}\\ (\underline{A}_2 \cap \underline{B}_2)(x) = \{&(\underline{A}_2 \cap \underline{B}_2)(1) = 1, (\underline{A}_2 \cap \underline{B}_2)(2) = 0.9,\\ &(\underline{A}_2 \cap \underline{B}_2)(3) = 0.6, (\underline{A}_2 \cap \underline{B}_2)(4) = 0.6\}\end{aligned}$$

From the structure of fuzzy sets $\underline{A}_1$ and $\underline{A}_2$, it is known that points 1 and 3 both belong to the core $A = \{1,3\}$ of the fuzzy sets; point 2 closer than point 4 to the core $A$. Although different membership function values are assigned to the points in $\underline{A}_1$ and $\underline{A}_2$ respectively, since the correlation among points are the same in the two fuzzy sets, their structure are the same. Therefore, the structural relation among points is the essential property of a fuzzy set.

In Definition 2.1 a fuzzy set is defined by a fuzzy equivalence relation so that the isomorphism principle holds. Otherwise, the isomorphism principle does not necessarily hold. For example, if a fuzzy set is defined from neighborhoods, we have the following example.

**Example 3.** Assume that the structures of fuzzy sets $\underline{A}$ and $\underline{B}$ are $\underline{A} = \{(1,2),(3),(4)\}$ and $\underline{B} = \{(1),(2),(3),(4)\}$, respectively. Define their membership functions as $A_1(x) = \{1,1,0.8,0.6\}$, $A_2(x) = \{1,1,0.8,0.7\}$, $B_1(x) = \{1,0.7,0.6,0.8\}$, and $B_2(x) = \{1,0.7,0.6,0.9\}$. Certainly, $A_1(x)$ and $A_2(x)$ are membership functions of $\underline{A}$, and $B_1(x)$ and $B_2(x)$ are membership functions of $\underline{B}$.

Performing the union operation over $A_1(x)$ and $B_1(x)$ ($A_2(x)$ and $B_2(x)$), we have

$(\underline{A} \cup \underline{B})_1(x) = \max\{A_1(x), B_1(x)\} = \{1, 1, 0.8, 0.8\}$, and the structure of $(\underline{A} \cup \underline{B})_1(x)$ is $\{(1,2),(3,4)\}$
$(\underline{A} \cup \underline{B})_2(x) = \max\{A_2(x), B_2(x)\} = \{1, 1, 0.8, 0.9\}$, and the structure of $(\underline{A} \cup \underline{B})_2(x)$ is $\{(1,2),(4),(3)\}$

Obviously, the two structures are not the same. The isomorphism principle does not hold.

## 3. The discriminance of isomorphism of fuzzy equivalence relations

In what conditions two fuzzy equivalence relations would be isomorphism?

**Theorem 3.1** (isomorphism discriminance theorem). *Two isomorphic fuzzy equivalence relations $R_1$ and $R_2 \Longleftrightarrow$ for any $x, y, u, v \in X, R_1(u,v) < R_1(x,y) \leftrightarrow R_2(u,v) < R_2(x,y)$ and $R_2(x,y) = R_2(u,v) \leftrightarrow R_1(x,y) = R_1(u,v)$.*

**Proof.** $\Rightarrow$: Assume $R_1(u,v) < R_1(x,y)$. Let $\lambda_1$:$R_1(u,v) < \lambda_1 < R_1(x,y)$ and $X_1(\lambda_1) = \{(x,y)|R_1(x,y) \geqslant \lambda_1\}$. Then, $x$ and $y$ are equivalent in $X_1(\lambda_1)$ but $u$ and $v$ are not equivalent in $X_1(\lambda_1)$. Since $R_1$ and $R_2$ are isomorphic, there exists an $\lambda_2$ such that $X_1(\lambda_1) = X_2(\lambda_2)$. We have $x$ and $y$ are equivalent in $X_2(\lambda_2)$ but $u$ and $v$ are not equivalent in $X_2(\lambda_2)$. Therefore, $R_2(u,v) < \lambda_2$ i.e., $R_2(u,v) < R_2(x,y)$.

Similarly, $R_2(u,v) < R_2(x,y) \to R_1(u,v) < R_1(x,y)$.

Now we prove that if $R_2(x,y) = R_2(u,v)$ then $R_1(x,y) = R_1(u,v)$. By reduction to absurdity, Let $R_2(x,y) = R_2(u,v)$ but $R_1(x,y) \neq R_1(u,v)$. We might as well assume $a = R_1(x,y) > R_1(u,v) = b$.

Letting $X_1(a) = \{(x,y)|R_1(x,y) \geqslant a\}$, then $(u,v) \notin X_1(a)$. From the isomorphism of $R_1$ and $R_2$, there exists an $c$ such that $X_1(a) = X_2(c)$. We have $(x,y) \in X_2(c) = X_1(a)$. And from $R_2(x,y) = R_2(u,v)$, we have $(u,v) \in X_2(c) = X_1(a)$ and $(u,v) \notin X_1(a)$. There is a contradiction. Thus, $R_1(x,y) = R_1(u,v)$.

$\Leftarrow$: Let $I_1 = \{\lambda | \exists (x,y), R_1(x,y) = \lambda,\ 0 \leqslant \lambda \leqslant 1\}$, and $\forall \lambda \in I_1$, let $D_1(\lambda) = \{(x,y)| R_1(x,y) = \lambda\}$. From the assumption $R_1(u,v) = R_1(x,y) \leftrightarrow R_2(u,v) = R_2(x,y)$, it is known that the value of $R_2$ of any point on $D_1(\lambda)$ has the same, e.g., $\mu$. We can define a function $f(\lambda) = \mu$ on $I_1$. For $\forall \lambda \in I_1$, letting $X_1(\lambda) = \{(x,y)|R_1(x,y) \geqslant \lambda\}$, we have a hierarchical structure $\{X_1(\lambda), \lambda \in I_1\}$. For $\forall \lambda \in I_1$, letting $X_2(f(\lambda)) = \{(x,y)| R_2(x,y) \geqslant f(\lambda)\}$, then have a hierarchical structure $\{X_2(f(\lambda)), \lambda \in I_1\}$.

Now we prove below that the hierarchical structures $\{X_1(\lambda), \lambda \in I_1\}$ and $\{X_2(f(\lambda)), \lambda \in I_1\}$ are the same. Given $X_1(\lambda) = \{(x,y)|R_1(x,y) \geqslant \lambda\}$, and assume $(u,v) \in X_1(\lambda)$, we have $R_1(u,v) \geqslant \lambda = R_1(x_1 y_1)$. On the other hand, from $R_1(u,v) \geqslant R_1(x,y) \leftrightarrow R_2(u,v) \geqslant R_2(x_1,y_1) = f(\lambda)$, we have $(u,v) \in X_2(f(\lambda))$.

Similarly, we have $(u,v) \in X_2(f(\lambda))$, and $(u,v) \in X_1(\lambda)$. That is, hierarchical structures $\{X_1(\lambda), \lambda \in I_1\}$ and $\{X_2(f(\lambda)), \lambda \in I_1\}$ are the same. $\square$

**Definition 3.1.** Given two fuzzy equivalence relations $R_1$ and $R_2$. If $\forall(x,y)$, $(u,v) \in X\times X, R_1(x,y) = R_1(u,v)$, $R_2(x,y) = R_2(u,v)$, then for $\forall(x,y) \in X\times X$, we define a function $F$:$F(R_1(x,y)) = R_2(x,y)$. If for any $(x,y),(u,v) \in X\times X$, when $R_1(x,y) < R_1(u,v)$, we have $F(R_1(x,y)) < F(R_1(u,v))$, i.e., $R_2(x,y) < R_2(u,v)$, and when $F(R_1(x,y)) < F(R_1(u,v))$, we have $R_1(x,y) < R_1(u,v)$. Function $F$ is called a strictly increasing function.

**Corollary 3.1.** *Given two fuzzy equivalence relations $R_1$ and $R_2$. Assume that $F$ is a function defined by Definition* 3.1. *If $F$ is a strictly increasing function, then $R_1$ and $R_2$ are isomorphic.*

**Proof.** From the definition of $F$ and Theorem 3.1, we can directly obtain the corollary. □

**Example 4.** Assume $X = R^m$. Let

$$R_1(x,y) = \mathrm{e}^{-\sum_k |x_k-y_k|}$$

$$R_2(x,y) = 1 - c\sum_k |x_k - y_k|$$

In the above formula, given a proper $c$ such that $0 \leqslant R_2(x,y) \leqslant 1$.

Obviously, there exists a strictly increasing function between $R_1$ and $R_2$. So $R_1$ and $R_2$ are isomorphic.

**Proposition 3.1.** *$R_1$ and $R_2$ are isomorphic $\Longleftrightarrow$ the corresponding function $F(R_1(x,y)) = R_2(x,y)$ is strictly increasing.*

**Proof.** $\Leftarrow$: From the Corollary 3.1, it can be proved.

$\Rightarrow$: Assume that $R_1$ and $R_2$ are isomorphic. From the proof process of Theorem 3.1, it is known that there exist $I_1$ and an one-one corresponding and increasing function $f$ such that $\{X_1(\lambda),\ \lambda \in I_1\}$ and $\{X_2(f(\lambda)),\ \lambda \in I_1\}$ are the same. Letting $F(\lambda) = f(\lambda)$, $\lambda \in I_1$, then $F(R_1(x,y)) = R_2(x,y)$ is a strictly increasing function. □

## 4. ε-Similarity of fuzzy sets

From the preceding discussion, it is known that the requirement of isomorphism of two fuzzy equivalence relations is too strong. We will present a rather week condition, i.e., ε-similarity of two fuzzy sets, and discuss its properties.

**Definition 4.1.** Given two fuzzy equivalence relations $R_1$ and $R_2$, and $\varepsilon > 0$, if there exists a fuzzy equivalence relation $R_3$ satisfying (1) $R_3$ and $R_1$ are strongly isomorphic, (2) $\forall x, y \in X$, $|R_2(x,y) - R_3(x,y)| < \varepsilon$; or (1) $R_3$ and $R_2$ are strongly isomorphic, (2) $\forall x, y \in X$, $|R_1(x,y) - R_3(x,y)| < \varepsilon$ then $R_1$ and $R_2$ are ε-similarity.

**Definition 4.2.** Given two fuzzy sets $\underline{A}$ and $\underline{B}$, and $\varepsilon > 0$, there exists a fuzzy set $\underline{C}$ satisfying (1) $\underline{A}$ and $\underline{C}$ are isomorphic (or $\underline{B}$ and $\underline{C}$ are isomorphic), (2) $\forall x \in X$, $|\mu_C(x) - \mu_B(x)| < \varepsilon$ (or $\forall x \in X$, $|\mu_C(x) - \mu_A(x)| < \varepsilon$), then $\underline{A}$ and $\underline{B}$ are ε-similarity.

**Proposition 4.1.** *Assume that $R_1$ and $R_2$ are two ε-similarity fuzzy equivalence relations, and $A$ is a crisp set on $X$. From $R_1$, $R_2$, and $A$, two fuzzy sets $\underline{A}$ and $\underline{B}$ can be defined by Definition* 2.1. *Fuzzy sets $\underline{A}$ and $\underline{B}$ are ε-similarity.*

**Proof.** We might as well assume there exists an $R_3$ strongly isomorphic to $R_1$, $R_3 = R_1$, and $\forall x, y \in X$, $|R_2(x,y) - R_3(x,y)|\varepsilon$. Given $x$, assuming that $A(x) = R_1(a_1, x)$ and $B(x) = R_2(a_2, x)$, then $A(x) = R_1(a_1, x) < R_2(a_1, x) + \varepsilon \leqslant R_2(a_2, x) + \varepsilon = B(x) + \varepsilon$.

Similarly, $B(x) = R_2(a_2, x) < R_1(a_2, x)\varepsilon \leqslant R_1(a_1, x) + \varepsilon = A(x) + \varepsilon$.

Thus, $|B(x) - A(x)| < \varepsilon$, and have $\underline{A}$ and $\underline{B}$ are ε-similarity. □

**Theorem 4.1** (Similarity principle). *$R_1$ and $R_2$ are two ε-similarity fuzzy equivalence relations on $X$. Given a family $\{A_1, A_2, \ldots, A_n\}$ of crisp sets. From $R_1$ and $R_2$, the families $A = \{\underline{A}_1, \ldots, \underline{A}_n\}$ and $B = \{\underline{B}_1, \ldots, \underline{B}_n\}$ of fuzzy sets can be defined, respectively. After a finite number of set operations* (*union*, *intersection*, *complement*), *we have a families $C = \{\underline{C}_1, \ldots, \underline{C}_m\}$ and $D = \{\underline{D}_1, \ldots, \underline{D}_m\}$ of fuzzy sets. Then, $C$ and $D$ are ε-similarity.*

**Proof.** Assume there exist strongly isomorphic fuzzy equivalence relations $R_3$ and $R_1$, and $\forall x, y \in X$, $|R_2(x,y) - R_3(x,y)| < \varepsilon$. Defining a family $E = \{\underline{E}_1, \ldots, \underline{E}_m\}$ of fuzzy sets from $R_3$, after a set of operations, we have a family $F = \{\underline{F}_1, \ldots, \underline{F}_m\}$ of fuzzy sets. From the week isomorphism principle, $C$ and $F$ are isomorphic.

On the other hand, from Proposition 4.1, $\forall i$, $|D_i(x) - F_i(x)| < \varepsilon$. From Definition 4.2, $D$ and $C$ are ε-similarity. □

The discriminance of ε-similarity of fuzzy sets.

**Theorem 4.2.** *$R_1$ and $R_2$ are ε-similarity $\Longleftrightarrow$ there exists a strictly increasing function F such that* $|F(R_1(x,y)) - R_2(x,y)| < \varepsilon$.

**Proof.** Let $R_3(x,y) = F(R_1(x,y))$. From Proposition 3.1, $R_3$ and $R_1$ are strongly isomorphic, From Definition 4.1, $R_1$ and $R_2$ are ε-similarity. □

The structural properties of ε-similarity of fuzzy equivalence relations.

Assume that $R_1$ and $R_2$ are two fuzzy equivalence relations. Their corresponding hierarchical structures are $\{X_1(\lambda)\}$ and $\{X_2(\mu)\}$, respectively. If $R_1$ and $R_2$ are ε-similarity, there exists a strictly increasing function $F$: $\lambda_1 = F(\lambda)$ such that $\forall\lambda$, $\exists\mu$, we have $X_2(\mu - \varepsilon) < X_1(F(\lambda)) < X_2(\mu + \varepsilon)$. Otherwise, $\forall\mu$, $\exists\lambda$, have $X_2(F(\lambda) - \varepsilon) < X_1(\mu) < X_2(F(\lambda) + \varepsilon)$.

The relationship can be indicated in Fig. 1.

Fig. 1 shows that although the corresponding hierarchical structures of $R_1$ and $R_2$ cannot be merged into one hierarchical structure, for any quotient space $X_1(\lambda_1)$ among $\{X_1(\lambda)\}$ there must exist quotient spaces $X_2(\mu_1 - \varepsilon)$ and $X_2(\mu_1 + \varepsilon)$ in $\{X_2(\mu)\}$ such that one is in front of $X_1(\lambda_1)$; the other is at the back of $X_1(\lambda_1)$. Contrarily, for any quotient space $X_2(\mu_1)$ among $\{X_2(\lambda)\}$, there must exist quotient spaces $X_1(\lambda_1 - \varepsilon)$ and $X_1(\lambda_1 + \varepsilon)$ in $\{X_1(\lambda)\}$ such that one is in front of $X_2(\mu_1)$; the other is at the back of $X_2(\mu_1)$.

We will show below that in Theorem 2.1 the conclusion still holds whereas the condition "$R_1$ and $R_2$ are strongly isomorphic" is replaced by "$R_1$ and $R_2$ are isomorphic". The outline of its proof is as follows.

From the proof process of Theorem 3.1, it is known that when $R_1$ and $R_2$ are isomorphic, there exist $I_1$ and an one-one corresponding and increasing function $f$ de-

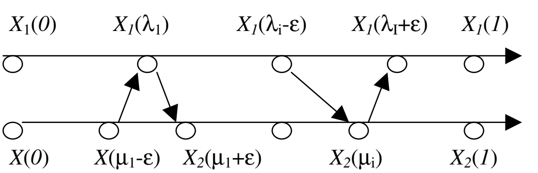


Fig. 1. The relationship among quotient spaces.

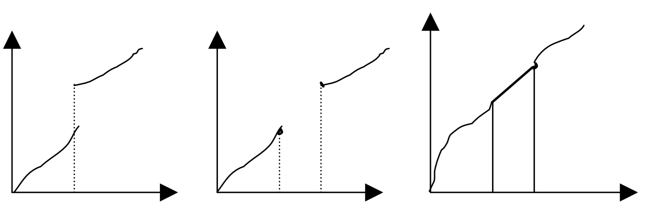

Fig. 2. The adjustment procedure of function $f$.

fined on $I_1$ such that $\{X_1(\lambda), \lambda \in I_1\}$ and $\{X_2(f(\lambda)), \lambda \in I_1\}$ are the same, i.e., $\forall \lambda \in I_1$, we have $X_1(\lambda) = X_2(f(\lambda))$. If $f$ can be extended to an one-one corresponding and increasing function from $[0, 1]$ on to $[0, 1]$, then $f$ must be continuous. Therefore, $R_1$ and $R_2$ are strongly isomorphic. Generally, the function extended from $f$ is not continuous on $[0, 1]$. When $X$ is a finite set, if the fuzzy equivalence relations on $X$ are isomorphic then they are strongly isomorphic as well.

We will show that as long as function $f$ is adjusted properly, it will become an one-one corresponding, continuous, and increasing function. The adjustment procedure of the function $f$ is as follows.

As shown in Fig. 2, along the discontinuous point we cut the curve $f$ vertically, move the right part of $f$ rightward, and connect the right and left parts by a line.

By using the above procedure to treat all discontinuous points of $f$ and $f^{-1}$ (the inverse of function $f$), we can obtain an one-one corresponding, continuous, and increasing function.

Therefore, the domain of $f$ is extended from $[0, 1]$ to $[0, \delta_1]$, since the discontinuous points of $f$ are moved rightward. The range of $f$ is extended to $[0, \delta_2]$, since the discontinuous points of $f^{-1}$ are moved rightward. Again, $\{X_1(\lambda), \lambda \in [0, 1]\}$ is extended to $\{X_3(\lambda)\} = \{X_1(\lambda), \lambda \in [0, \delta_1]\}$. It is easy to prove that the equivalence relations $R_1$ and $R_3$ corresponding to $\{X_1(\lambda)\}$ and $\{X_3(\lambda)\}$ are ε-similarity as long as the overall rightward moving distance of discontinuous points in $f$ is less than ε.

Since the adjusted function is one-one corresponding and continuous from $[0, \delta_1]$ to $[0, \delta_2]$, we can obtain that the equivalence relations $R_3$ and $R_4$ corresponding to $\{X_3(\lambda)\}$ and $\{X_4(\lambda)\}$ are strongly isomorphic.

Then, we have the following theorem.

**Theorem 4.3** (isomorphism principle). *$R_1$ and $R_2$ are two isomorphic fuzzy equivalence relations on X. Given a family $\{A_1, A_2, \ldots, A_n\}$ of crisp sets. From $R_1$ and $R_2$, the families $A = \{\underline{A}_1, \ldots, \underline{A}_n\}$ and $B = \{\underline{B}_1, \ldots, \underline{B}_n\}$ of fuzzy subsets can be defined. After performing a finite number of set operations (complement, intersection, union, etc.) over A and B, we have new families $C = \{\underline{C}_1, \ldots, \underline{C}_m\}$ and $D = \{\underline{D}_1, \ldots, \underline{D}_m\}$ of fuzzy subsets. C and D are isomorphic as well.*

**Proof.** Assume that $\{X_i(\lambda)\}$ corresponds to fuzzy equivalence relation $R_i$, $A^3 = \{\underline{A^3}_1, \ldots, \underline{A^3}_n\}$ is a family of fuzzy sets defined by $R_3$ and a family $A = \{\underline{A}_1, \ldots, \underline{A}_n\}$ of crisp sets. After a set of operations over $A^3$, we have a family $C^3$ of fuzzy sets. $A^4 = \{\underline{A^4}_1, \ldots, \underline{A^4}_n\}$ is a family of fuzzy sets defined by $R_4$ and a family $A = \{\underline{A}_1, \ldots, \underline{A}_n\}$ of crisp sets. After a set of operations over $A^4$, we have a family $D^4$ of fuzzy sets. Since $R_1$ and $R_3$ are ε-similarity, from Theorem 4.2 the families $C^3$ and $C$ of fuzzy sets are ε-similarity. Similarly, since $R_2$ and $R_4$ are ε-similarity, the families $D^4$ and $D$ of fuzzy sets are ε-similarity.

Again, $R_3$ and $R_4$ are strongly isomorphic, from Theorem 2.1 $C^3$ and $D^4$ are isomorphic.

Finally, $C$ and$D$ are 2ε-similarity. Since ε is an arbitrary number, $C$ and $D$ are isomorphic. □

*Note:* From Theorem 4.3, in Definition 4.1, the "strongly isomorphic" condition can also be replaced by "isomorphic".

## 5. The geometric meaning of the structural definition of fuzzy sets

Since the membership functions (see Section 2) of fuzzy sets are induced from fuzzy equivalence relations, the geometric meaning of fuzzy set structures will be analyzed by the fuzzy equivalence relation structures.

Given a fuzzy equivalence relation $R(x,y)$. First, assume that $A = \{x_0\}$ is a single point set. A fuzzy set $\underline{A}$ is generated from $A$ and its membership function $A(x) = R(x_0,x)$ is defined from $\{x_0\}$. From Definition 2.1, letting $d(x,y) = 1 - R\ (x,y)$, $d(x,y)$ is a normalized equicrural distance on some quotient space $[X]$ of $X$. Under the distance, the neighborhood system of $x_0$ is $\{S(x_0,\varepsilon),\ 0 \leqslant \varepsilon \leqslant 1\}$, where $S(x_0,\varepsilon) = \{x|d(x_0,x) < \varepsilon, x \in X\}$ corresponds to the structure of fuzzy set $\underline{A}(A(x_0))$.

A totally ordered quotient space $([X_A], <)$ can be defined from the neighborhood system by Definition 2.3. The distance between any point and point $x_0$ determines its order on the space.

Generally, given a crisp set $A$, from Definition 2.1 a fuzzy set $\underline{A}$ can be defined and its membership function is $A(x)$. Letting $d(A,x) = 1 - A(x)$, $d(A,x)$ is the distance between point $x$ and set $A$ under metric $d(x,y)$. Let $S(A,\varepsilon) = \{x|d(A,x) < \varepsilon\}$ be an $\varepsilon$-neighborhood of $A$. From the neighborhood system $\{S(A,\varepsilon),\ 0 \leqslant \varepsilon \leqslant 1\}$ of $A$, a totally ordered quotient space can also be obtained by Definition 2.3, where the distance between any point and set $A$ determines the space order. From the isomorphism principle in Section 4, it is known that the corresponding totally ordered quotient spaces of fuzzy subsets represent the essential property of fuzzy subsets.

A crisp set and a corresponding fuzzy set can be described geometrically as in Fig. 3.

The geometrical meaning of the structural definition of fuzzy sets.

According to Definition 2.3, given a fuzzy set $\underline{A}$ on $X$, an order relation on some quotient space $[X]$ of $X$ is obtained. The order represents the relative distance, a well relation, between each point on $X$ and the core $A_0$ of fuzzy set $\underline{A}$ rather than an absolute value as described by common membership functions. So the structural definition of fuzzy sets are better than the optionally chosen membership functions,

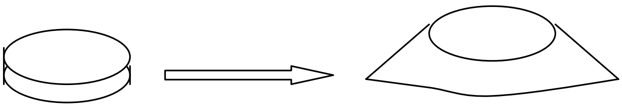


Fig. 3. The geometric representation of a common set and a fuzzy set.

since the relative order relation among elements (points) on $X$ is the essential property of a fuzzy set.

From the preceding discussion we know that as long as people have the same (or $\varepsilon$-similarity) interpretation of the order relation among points of a fuzzy set on $X$, the membership functions defined by Definition 2.1 are isomorphic (or $\varepsilon$-similarity), although their types may (slightly) be different. From the isomorphism ($\varepsilon$-similarity) principle, after a finite number of set operations the fuzzy sets obtained are also isomorphic (or $\varepsilon$-similarity). Under a certain environment, a group of persons generally has the same (or similar) structural interpretation for a concept so that the fuzzy processing is more robust.

**Example 5.** As in Example 1, Let $A = \{1\}$, two isomorphic fuzzy equivalence relations $R_1$ and $R_2$.

Defining fuzzy set $\underline{A}$ from $R_1$, we have its membership function.

$A_1(x) = \{A_1(1) = 1, A_1(2) = 0.8, A_1(3) = 9.5, A_1(4) = 0.5\}$, and the corresponding totally ordered quotient space $\{(1), (1,2), (1,2,3,4)\}$.

We might as well define $\underline{A}$ from $R_2$ and obtain

$A_2(x) = \{A_2(1) = 1,\ A_2(2) = 0.9,\ A_2(3) = 0.6,\ A_2(4) = 0.6\}$, and its corresponding totally ordered quotient space $\{(1), (1,2), (1,2,3,4)\}$.

Two membership functions $A_1(x)$ and $A_2(x)$ are (slightly) different, but in quotient space $[X]_A$ they have the same three elements (1), (2), and (3,4) and the same order relation among them, i.e., $(1) < (2) < (3,4)$. The fuzzy set $\underline{A}$ defined by $A_1(x)$ and $A_2(x)$ respectively has the same structure.

## 6. Conclusions

By introducing the fuzzy equivalence relation into quotient space theory, we present a structural definition of fuzzy sets. Then the "isomorphism" and "similarity" principles of fuzzy sets are given. From the principles, we show that although each person may probably assign (slightly) different membership functions for a concept in problem solving, the same (approximate) results can be obtained by fuzzy processing generally.

The results will help us for deeper understanding the essence of fuzzy processing.

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## References

[1] L.A. Zadeh, Fuzzy sets, Information and Control 3 (8) (1965) 338–353.
[2] T.Y. Lin, Granular fuzzy sets: a view from rough set and probability theories, International Journal of Fuzzy Systems 3 (2) (2001) 373–381.
[3] T.Y. Lin, Sets with partial memberships: a rough sets view of fuzzy sets, in: 1998 World Congress of Computational Intelligence (WCCI98), Anchorage, Alaska, May 4–9, 1998, pp. 785–790.
[4] P. Liang, F. Song, What does a probabilistic interpretation of fuzzy sets mean? IEEE Transactions on Fuzzy Systems 4 (2) (1996) 200–205.
[5] T. Mitsuishi, N. Endou, Y. Shidama, The concept of fuzzy set and membership function and basic properties of fuzzy set operation, Journal of Formalized.
[6] T.Y. Lin, Neighborhood systems and approximation in relational databases and knowledge bases, in: Proceedings of the 4th International Symposium on Methodologies of Intel Intelligent Systems, 1988.
[7] T.Y. Lin, Neighborhood systems—application to qualitative fuzzy and rough sets, in: P.P. Wang (Ed.), Advances in Machine Intelligence and Soft-Computing, Department of Electrical Engineering, Duke University, Durham, North Carolina, USA, 1997, pp. 132–155.
[8] T.Y. Lin, Topological and fuzzy rough sets, in: R. Slowinski (Ed.), Decision Support by Experience—Application of the Rough Sets Theory, Kluwer Academic Publishers, 1992, pp. 287–304.
[9] T.Y. Lin, Granular fuzzy sets: crisp representation of fuzzy sets, in: European Congress on Intelligent Techniques and Soft Computing, September 7–10, 1998, pp. 94–98.
[10] T.Y. Lin, A set theory for soft computing, in: Proceedings of 1996 IEEE International Conference on Fuzzy Systems, New Orleans, Louisiana, September 8–11, 1996, pp. 1140–1146.
[11] T.Y. Lin, S. Tsumoto, Qualitative fuzzy sets revisited: granulation on the space of membership functions, in: The 19th International Meeting of the North American Fuzzy Information Processing Society, July 1–15, 2000, Atlanta, pp. 331–337.
[12] T.Y. Lin, Qualitative fuzzy sets: a comparison of three approaches, in: Proceeding of Joint 9th IFSA World Congress and 20th NAFIPS International Conference, Vancouver, Canada, July 25–28, 2001, pp. 2359–2363.
[13] B. Zhang, L. Zhang, The Theory and Applications of Problem Solving, Elsevier Science Publishers, North-Holland, 1992.
[14] L. Zhang, B. Zhang, Fuzzy quotient space theory (fuzzy granule computing methods), Chinese Journal of Software 14 (4) (2003) 770–776.
[15] L. Zhang, B. Zhang, Quotient space theory of problem solving, Fundamenta Informaticae 59 (2004) 1–12, IOS Press, Poland.
[16] Y.Y. Yao, X.C. Chen, Neighborhood based information systems, in: Proceedings of the 3rd Joint Conference on Information Sciences, Rough Set & Computer Science Research Triangle Park, North Carolina, USA, March 1–5 1997, Vol. 3, pp. 154–157.